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The stochastic quantisation of U(N) and SU(N) lattice gauge theory and Langevin equations for the Wilson loops

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Abstract. We perform the stochastic quantisation of U(N) and SU(N) lattice gauge theories. For $N = 1$ and 2 we do this by studying the stochastic motion on the circle and the sphere S^3 while the generalisation for any N is achieved by imposing the unitarity constraints by means of Lagrange multipliers. We also find the Langevin equations for the Wilson loops and show that when averaged over the random forces and the new time dimension is taken to infinity they become the Schwinger–Dyson equations of the corresponding gauge theory.

1. Introduction

The stochastic quantisation of quantum field theories (Parisi and Wu 1981) has motivated a great deal of interest not only because it provides us with a novel quantisation procedure by also because of its applications in the numerical simulation of lattice gauge theories (Hamber and Parisi 1983) (LGT).

An extra time dimension is added to the quantum field theory (in Euclidean metric) and it is treated as a non-equilibrium statistical mechanical problem. In a way this is similar to what occurs in the usual Monte Carlo simulation of a lattice system where the new dimension is represented by the time it takes to reach the equilibrium distribution. The dependence of the fields on this fifth coordinate is determined by Langevin equations with stochastic forces following Gaussian distributions; the quantum field theory results are obtained only when the system reaches its equilibrium distribution as $t \to \infty$. Alternatively Fokker–Planck equations for the probability distribution could be written down directly (Parisi and Wu 1981, Floratos and Iliopoulos 1982).

Standard partition function results are sometimes easier to obtain in this new formalism: an interesting example was given by Alfaro and Sakita (1983) and Aldazabal *et al* (1983b) in the derivation of reduced models.

Parisi and Wu applied this quantisation procedure to gauge theories by proposing Langevin equations for the potential. However an interesting alternative is to consider the stochastic equations satisfied by the gauge invariant quantities of the theory. This was done in a previous paper (Aldazabal *et al* 1983a) for the Abelian compact LGT. An advantage of this idea is that the equivalence with the standard partition function approach can be easily established at all orders in the strong coupling expansion by showing that the Langevin equations for the Wilson loops become the Schwinger–Dyson equations of the corresponding LGT as $t \to \infty$.

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In this paper we report about the generalisation of these results to the non-Abelian unitary groups U(N) and SU(N). Langevin equations can be easily given when all the fields are independent variables; this is not the case however for the groups under consideration because they have to satisfy the unitarity constraints. An immediate solution to this problem might be to introduce a parametrisation for the group and after that to write Langevin equations for the independent fields only. This approach works out well for the Abelian case and for $N=2$ but it is rather difficult to extend it to larger groups.

A way out of this problem is to consider the elements of the unitary matrix as independent variables and to take care of the constraints by means of Lagrange multipliers. As we shall see later on, this turns out to be a convenient tool for the stochastic quantisation of LGT.

This paper is organised as follows. In § 2 we present our stochastic quantisation of Abelian LGT in D dimensions. A Langevin equation is written down for the angle which parametrises the circle S^1 and from it we obtain the corresponding equations for the link variables and the Wilson loops. We also show how to do strong coupling expansions with this formalism. Finally the equivalence between the Langevin equations for the loops (once they are averaged over the random forces and the extra time is sent to infinity) and the Schwinger-Dyson equations are established.

In the next section we study the non-Abelian U(2) and SU(2) LGT by parametrising the corresponding group manifolds $S^1 \times S^3$ and S^3 respectively. We also show that for the one-plaquette problem the eigenvalues of the U(2) matrix behave as a time dependent Coulomb gas on a circle in the presence of an external field.

In § 3 we perform the quantisation of U(N) LGT for any N and in D dimensions by using Lagrange multipliers. As in § 2 we give a simple example of strong coupling perturbation theory and establish the connection with the Schwinger-Dyson equations. We generalise these results to SU(N) in the last section. Some technical aspects are discussed in three appendices.

When this work was under completion we received two preprints (Alfara and Sakita 1982, Guha and Lee 1982) which also deal with the stochastic quantisation of U(N) LGT. Although part of the results coincide, our approach is completely different from theirs.

2. Stochastic quantisation of U(1) LGT

We begin by treating the simplest case, U(1) compact QED (Floratos and Iliopoulos 1982, Zwanziger 1981). In the usual functional approach (Wilson 1974) one works with the partition function

$$Z=\int \Pi_l \, \mathrm{d}\theta_l \, \mathrm{e}^{-S} \tag{2.1}$$

with

$$S=-\beta \sum_P \cos \theta_P \tag{2.2}$$

where l denotes the links and P the plaquettes. Operator vacuum expectation values are obtained from

$$\langle O\rangle_S = \int \Pi_l \, d\theta_l 0 \, e^{-S}. \tag{2.3}$$

In the stochastic quantisation the independent angles θ_l acquire their dependence in the new time dimension t by means of a Langevin equation. We propose

$$\frac{\partial \theta_l(t)}{\partial t} = -\beta \sum_{\{P_l\}} \sin \theta_{P_l}(t) + \eta_l(t) \tag{2.4}$$

where $\{P_l\}$ is the set of all the plaquettes attached to the link l and $\eta_l(t)$ is a Gaussian stochastic force defined at that link†.

In this new formalism mean values are calculated by averaging over the Gaussian random force. An operator 0 now gets its dependence on t through the $\theta_l(t)$ and

$$\langle 0(t)\rangle_\eta = \frac{1}{z}\int D\eta_l(t) 0(t) \tag{2.5}$$

where

$$D\eta_l(t) = \left(\prod_{l,t} d\eta_l(t)\right) \exp\left(-\frac{1}{4}\int_0^\infty \eta_l^2(t')\, dt'\right) \tag{2.6}$$

and

$$z = \int D\eta_l(t). \tag{2.7}$$

The quantum mechanical results are supposedly obtained by taking the limit $t \to \infty$, namely

$$\lim_{t\to\infty} \langle 0(t)\rangle_\eta = \langle 0\rangle_S. \tag{2.8}$$

What we shall do here is to show that this is true for the Langevin equation (2.4). The basic idea is to derive stochastic equations for gauge invariant quantities (the Wilson loops) and to show that when the large time limit is taken they become the Schwinger–Dyson equations of lattice QED (Foerster 1979, Eguchi 1979, Weingarten 1979).

From (2.4) we see that the link variables

$$U(l, t) = \exp(i\theta_l(t)) \tag{2.9}$$

satisfy

$$\frac{\partial U(l,t)}{\partial t} = -\frac{\beta}{2} U(l,t) \sum_{\{P_l\}} [U_{P_l}(t) - U^*_{P_l}(t)] + i\eta_l(t) U(l,t). \tag{2.10}$$

Here U_P is the product of the U_l's around the plaquette P. For the Wilson loop $W_c(t) = \Pi_{l\in C}\, U_l(t)$, using (2.10) we have

$$\frac{\partial W_c(t)}{\partial t} = -\frac{\beta}{2} W_c(t) \sum_{\substack{l\in C\\ \{P_l\}}} [U_{P_l}(t) - U^*_{P_l}(t)] + i\eta_c(t) W_c(t) \tag{2.11}$$

† As is well known this expression is only a pre-equation. To be turned into an actual equation it must be supplemented with an additional interpretation rule. Calculations that follow are consistent with the Stratonovich prescription (van Kampen 1980).

where

$$\eta_c(t)=\sum_{l\in C}\eta_l(t). \tag{2.12}$$

In order to obtain the solutions of these equations in the form of a strong coupling expansion we first propose

$$W_c(t)=\exp\left(\mathrm{i}\int_0^t \eta_c(t')\,\mathrm{d}t'\right)V_c(t), \tag{2.13}$$

the first factor is the solution of (2.11) for $\beta=0$ and the expansion for $V_c(t)$ is

$$V_c(t)=\sum_{n=0}^{\infty}(\beta/2)^n v_c^{(n)}(t) \tag{2.14}$$

with

$$v_c^{(0)}(t)=1. \tag{2.15}$$

From (2.11)-(2.15) we have the nth order of the Wilson loop

$$W_c^{(n)}(t)=-W_c^{(0)}(t)\sum_{k=0}^{n-1}\int_0^t \mathrm{d}t'\,v_c^{(k)}(t')\sum_{\substack{l\in C\\ \{P_l\}}}[U_{P_l}^{(n-1-k)}(t)-U_{P_l}^{(n-1-k)}(t)]. \tag{2.16}$$

As a trivial example we show how to evaluate the first two orders of $U_P(t)$. The first one is

$$\langle U_P^{(0)}(t)\rangle_\eta=\frac{1}{z}\int \mathrm{D}\eta_l(t)\exp\left(\mathrm{i}\int_0^t \eta_P(t')\,\mathrm{d}t'\right). \tag{2.17}$$

Here we observe that times in the interval $[t,\infty]$ contribute with a factor one while times in $[0,t]$ give an $\exp(-t)$ for each of the links around the plaquette P, then

$$\langle U_P^{(0)}(t')\rangle_\eta=\mathrm{e}^{-4t}\to 0 \qquad \text{as } t\to\infty. \tag{2.18}$$

Using (2.16) the next order is

$$\langle U_P^{(1)}(t)\rangle_\eta=-\frac{\beta}{2z}\int \mathrm{D}\eta_l(t)\exp\left(\mathrm{i}\int_0^t \eta_P(t')\,\mathrm{d}t'\right)$$
$$\times\int_0^t \mathrm{d}t'\sum_{\substack{l\in P\\ \{P_l\}}}\left[\exp\left(\mathrm{i}\int_0^{t'}\eta_{P_l}(t'')\,\mathrm{d}t''\right)-\exp\left(-\mathrm{i}\int_0^{t'}\eta_{P_l}(t'')\,\mathrm{d}t''\right)\right]. \tag{2.19}$$

The reason why (2.18) is zero as $t\to\infty$ is that in (2.17) there is a phase contributing at all times from 0 to t. In (2.19), however, it is possible to obtain a non-zero result by cancelling the first phase by those inside the sum over plaquettes. This corresponds to choosing only the plaquette P in the set $\{P_l\}$ and the last term of that sum. Since P appears four times we have

$$\langle U_P^{(1)}(t)\rangle_\eta=\frac{\beta}{2}4\int_0^t \mathrm{d}t'\exp[-4(t-t')]$$
$$=\beta/2 \qquad \text{as } t\to\infty. \tag{2.20}$$

A similar reasoning allows to evaluate higher orders in the limit of large t.

In order to relate the Langevin equation (2.11) with the Schwinger–Dyson equations we first prove that at all orders in perturbation theory

$$\langle W_c(t)\eta_l(t)\rangle_\eta = \mathrm{i}\langle W_c(t)\rangle_\eta \tag{2.21}$$

where l belongs to C. This is true because at a given order η a typical term on the left-hand side is of the form

$$\frac{1}{z}\left(\frac{\beta}{2}\right)^n \int \mathrm{D}\eta_l(t)\exp\left(\mathrm{i}\int_0^t n_c(t')\,\mathrm{d}t'\right)\eta_l(t)\int_0^t \mathrm{d}t'\, U^{(0)}_{P_1}(t')$$
$$\times\ldots\int_0^t \mathrm{d}t''\, U^{(0)}_{P_2}(t'')\int_0^{t^{(n-1)}} \mathrm{d}t^{(n)}\, U^{(0)}_{P}(t^{(t)}) \tag{2.22}$$

and except for a finite number of points the integral over $\eta_l(t)$ is

$$\int_{-\infty}^{\infty} \mathrm{d}\eta_l(t)\exp[-\tfrac{1}{4}\eta_l^2(t)\Delta t+\mathrm{i}\Delta t/2\eta_l(t)]\eta_l(t)$$
$$\times\left(\int_{-\infty}^{\infty}\mathrm{d}\eta_l(t)\exp[-\tfrac{1}{4}\eta_l^2(t)\Delta t]\right)^{-1}=\mathrm{i} \tag{2.23}$$

and the relation (2.21) is verified†.

Averaging over $\eta(t)$ in (2.11) and using (2.21) we obtain (L is the perimeter of the loop C)

$$\left\langle\frac{\partial W_c(t)}{\partial t}\right\rangle_\eta = -\frac{\beta}{2}\sum_{\substack{l\in C\\ \{P_l\}}}[\langle W_c(t)(U_{P_l}-U^*_{P_l})\rangle_\eta] - L\langle W_c(t)\rangle_\eta \tag{2.24}$$

and since as $t\to\infty$ the left-hand side averages to zero

$$\lim_{t\to\infty} L\langle W_c(t)\rangle_\eta = \frac{\beta}{2}\lim_{t\to\infty}\sum_{\substack{l\in C\\ \{P_l\}}}[\langle W_c(t)U^*_{P_l}(t)\rangle_\eta - \langle W_c(t)U_{P_l}(t)\rangle_\eta]. \tag{2.25}$$

The first (second) term in the sum corresponds to a loop obtained from C by adding on the link l a new plaquette with an opposite (the same) orientation to that of C.

We recognise in (2.25) the Schwinger–Dyson equation of QED on a lattice (Foerster 1979, Eguchi 1979, Weingarten 1979).

If the loop C is such that a given link l is traversed twice we have to modify (2.21). In this case we have

$\langle W_c(t)\eta_l(t)\rangle_\eta$

$= 2\mathrm{i}\langle W_c(t)\rangle_\eta$ if the link is traversed in the same direction

$= 0$ if it is traversed in the opposite direction.

But this is also in agreement with the Schwinger–Dyson equations for this kind of loop.

This completes our proof that the Langevin equation for the Wilson loops, (2.11), reduces to the Schwinger–Dyson equations giving a correct stochastic quantisation of lattice QED.

† From here we see that (2.21) is not well defined since the limits from above and below are different. However this is a particular case of the well known result $\langle\eta(t)g(x)\rangle_\eta = \langle g(x)\,\mathrm{d}g(x)/\mathrm{d}x\rangle_\eta$ where x satisfies the stochastic equation $x = f(x)+g(x)\eta(t)$ in the sense of Stratonovich. We gratefully acknowledge G Parisi for a comment on this point.

3. Stochastic motion on $S^1 \times S^3$ and S^3

The generalisation of the results of the previous section to the groups U(2) and SU(2) is straightforward. To avoid unnecessary complications we shall deal only with the one-plaquette gauge theory postponing the generalisation to any number of dimensions (and arbitrary N) to the last two sections. Then the action is

$$S = -\tfrac{1}{2}\beta(\mathrm{Tr}\, U + \mathrm{Tr}\, U^{+}) \tag{3.1}$$

where U belongs to U(2) or SU(2). We start by writing down the general form of an element of U(2)†

$$U = \gamma\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha^* \end{pmatrix} \tag{3.2a}$$

$$= \gamma \cdot u \tag{3.2b}$$

where

$$|\gamma|^2 = 1 \qquad |\alpha|^2 + |\beta|^2 = 1 \tag{3.3a, b}$$

then γ belongs to U(1) and u to SU(2).

We can think of the group manifold embedded in R^6 with cartesian coordinates $x^{\nu}{}_{,\nu} = 1, \ldots, 6$ such that

$$\gamma = x^6 + \mathrm{i}x^5 \qquad \alpha = x^4 + \mathrm{i}x^3 \qquad \beta = x^2 + \mathrm{i}x^1. \tag{3.4}$$

A parametrisation satisfying (3.4) is

$$\begin{aligned} x^1 &= \sin\theta_2 \sin\theta_1 \cos\psi \qquad & x^2 &= \sin\theta_2 \sin\theta_1 \sin\psi \\ x^3 &= \sin\theta_2 \cos\theta_1 & x_4 &= \cos\theta_2 \end{aligned} \tag{3.5a}$$

for the subgroup SU(2) and

$$x^5 = \sin\theta \qquad x^6 = \cos\theta \tag{3.5b}$$

for the U(1).

With this parametrisation the action (3.1) reads

$$S = -2\beta \cos\theta \cos\theta_2. \tag{3.6}$$

The simplest Langevin equations we can write for the $x^{\nu}(t)$ are

$$\dot{x}^{\nu} = -\delta S/\delta x_{\nu} + \xi^{\nu}(t) \tag{3.7}$$

where the $\xi^{\nu}(t)$ are a set of six stochastic forces following a Gaussian distribution. The mean value of a stochastic variable $A(\xi^{\nu})$ is given by

$$\langle A(\xi^{\nu})\rangle_{\xi} = \frac{1}{z}\int \prod_{\nu=1}^{6} \mathrm{D}\xi^{\nu}(t) A(\xi^{\nu}) \tag{3.8}$$

with

$$D\xi^{\nu}(t) = \prod_{t} \mathrm{d}\xi^{\nu}(t) \exp\left(-\tfrac{1}{4}\int_0^{\infty} \xi^{\nu^2}(t')\,\mathrm{d}t'\right) \tag{3.9}$$

† Strictly speaking $U(2) = SU(2) \times U(1)/Z_2$, however since we are studying diffusion on the group manifold, which is a local process, we can take $U(2) = SU(2) \times U(1)$.

and

$$z=\int\prod_{\nu=1}^{6}\mathrm{D}\xi^{\nu}(t). \tag{3.10}$$

Equations (3.7) do not take into account the constraints (3.3a, b). In order to implement them it is convenient to change to polar coordinates (a, θ) in the plane (x^5, x^6) and to spherical ones $(r, \psi, \theta_1, \theta_2)$ in the extra R^4. To do this we follow Graham (1977) (see also appendix 1) obtaining (the η's are also Gaussian stochastic forces with the same variance as the ξ^ν)

$$\begin{aligned}\dot a&=(\partial/\partial a)(-S+\ln a)+\eta_a\\ \dot\theta&=a^{-1}(-\partial S/\partial\theta+\eta_\theta)\end{aligned} \tag{3.11}$$

for R_2 and

$$\begin{aligned}\dot r&=(\partial/\partial r)(-S+3\ln r)+\eta_r\\ \dot\psi&=\frac{1}{r\sin\theta_1\sin\theta_2}\left(-\frac{\partial S}{\partial\psi}+\eta_\psi\right)\\ \dot\theta_1&=\frac{1}{r\sin\theta_2}\left[\frac{\partial}{\partial\theta_1}\left(-S+\frac{\ln\sin\theta_1}{r\sin\theta_2}\right)+\eta_1\right]\\ \dot\theta_2&=\frac{1}{r}\left[\frac{\partial}{\partial\theta_2}\left(-S+\frac{\ln\sin^2\theta_2}{r}\right)+\eta_2\right]\end{aligned} \tag{3.12}$$

for R^4. Notice that the factors in front of the brackets are the corresponding arc elements in curvilinear coordinates.

Since we are interested in the stochastic motion on the manifold $S^1\times S^3$ we have to enforce

$$a(t)=r(t)=1. \tag{3.13}$$

This can be easily achieved by using Lagrange multipliers which transform the stochastic equations for $\dot r$ and $\dot a$ into deterministic ones with the time independent solutions (3.13). Once this is done our set of stochastic equations become

$$\dot\theta=-2\beta\sin\theta\cos\theta_2+\eta_\theta \tag{3.14}$$

and

$$\dot\psi=\eta_\psi/(\sin\theta_1\sin\theta_2) \tag{3.15a}$$

$$\dot\theta_1=\cot\theta_1/(\sin^2\theta_2)+\eta_1/\sin\theta_2 \tag{3.15b}$$

$$\dot\theta_2=-2\beta\cos\theta\sin\theta_2+2\cot\theta_2+\eta_2. \tag{3.15c}$$

To obtain the Langevin equation for the plaquette variable U we have to go back to (3.2b) and the parametrisation (3.5) from where

$$\begin{aligned}\partial\cos\theta_2/\partial t&=\mathrm{Re}\,\dot u_{11}\\ \partial\sin\theta_2\cos\theta_1/\partial t&=\mathrm{Im}\,\dot u_{11}\\ \partial\sin\theta_2\sin\theta_1\sin\psi/\partial t&=\mathrm{Re}\,\dot u_{12}\\ \partial\sin\theta_2\sin\theta_1\cos\psi/\partial t&=\mathrm{Im}\,\dot u_{12}\end{aligned} \tag{3.16}$$

and

$$\dot{U}_{ij} = (\partial/\partial t)(e^{i\theta} u_{ij}). \tag{3.17}$$

Using here the equations for $\dot{\theta}$, $\dot{\psi}$, $\dot{\theta}_1$ and $\dot{\theta}_2$ we have the matrix equation

$$\partial U(t)/\partial t = -\beta(U^2 - I) + iH(t)U(t) \tag{3.18}$$

where H is a Hermitian stochastic matrix. To the interested reader in following the (not completely trivial) steps from (3.14) and (3.15) to (3.18) we refer to appendix 1 where we also describe the relation between the η's and the matrix H. In particular we show that

$$\eta_\theta = \tfrac{1}{2} \operatorname{Tr} H \tag{3.19}$$

and that the mean values have to be taken with the measure

$$DH(t) = \prod_t \prod_{i=1,2} dH_{ii}(t)\, dH_{12}\, dH_{12}^* \exp\left(-\tfrac{1}{8}\int_0^\infty \operatorname{Tr} H^2(t')\, dt'\right). \tag{3.20}$$

To obtain the Langevin equations for the group SU(2) it is sufficient to realise that the condition $\det U(t) = 1$ implies $\theta = 0$ at all times, then from (3.14) and (3.19) we see that the averaging has to be done with Hermitian matrices of zero trace. However U still satisfies (3.18).

Let us remark that the measure (3.20) is invariant under transformations of the form

$$H(t) \to \Omega(t)\eta(t)\Omega^+(t) \tag{3.21}$$

with Ω an element of U(N). Since the integral over Ω is trivial (see for instance appendix 2) the only relevant variables are the eigenvalues of $H(t)$. Then we are led to consider the stochastic equations for the eigenvalues $\lambda_1 = e^{i\alpha_1}$ and $\lambda_2 = e^{i\alpha_2}$ of the unitary matrix. In order to obtain them we first realise that from (3.2)-(3.5)

$$\begin{aligned}\det U(t) &= e^{2i\theta}\\ &= \exp[i(\alpha_1 + \alpha_2)]\end{aligned}$$

then

$$\theta = \tfrac{1}{2}(\alpha_1 + \alpha_2). \tag{3.23}$$

Besides

$$\begin{aligned}\operatorname{Tr} U(t) &= 2\cos\theta_2\, e^{i\theta}\\ &= e^{i\alpha_1} + e^{i\alpha_2}\end{aligned}$$

which yields

$$\theta_2 = \tfrac{1}{2}(\alpha_1 + \alpha_2). \tag{3.24}$$

Using (3.14) and (3.15a) we have

$$\begin{aligned}\dot{\alpha}_1 &= \dot{\theta} + \dot{\theta}_2 = -\beta \sin\alpha_1 + 2\cot\tfrac{1}{2}(\alpha_1 - \alpha_2) + \delta_1\\ \dot{\alpha}_2 &= \dot{\theta} - \dot{\theta}_2 = -\beta \sin\alpha_2 - 2\cot\tfrac{1}{2}(\alpha_1 - \alpha_2) + \delta_2\end{aligned} \tag{3.25}$$

with δ_j two Gaussian stochastic forces.

It is worthwhile writing this in a slightly different way by noticing that the action can be expressed as

$$S = -\beta(\cos\alpha_1 + \cos\alpha_2) \tag{3.26}$$

and that

$$\cot\tfrac{1}{2}(\alpha_1 - \alpha_2) = 2(\partial/\partial\alpha_1)\ln|e^{i\alpha_1} - e^{i\alpha_2}| \tag{3.27}$$

then

$$\begin{aligned}\dot{\alpha}_1 &= (\partial/\partial\alpha_1)(-S + 4\ln|e^{i\alpha_1} - e^{i\alpha_2}|) + \delta_1 \\ \dot{\alpha}_2 &= (\partial/\partial\alpha_2)(-S + 4\ln|e^{i\alpha_1} - e^{i\alpha_2}|) + \delta_2.\end{aligned} \tag{3.28}$$

These two equations describe the stochastic motion of two charges moving on a circle interacting by means of a Coulomb potential in the presence of an external field. This is in agreement with the well known result in the partition function approach that for the U(N) one-plaquette LGT the eigenvalues of the unitary matrix behave as a Coulomb gas on a circle (Gross and Witten 1980).

4. U(N) lattice gauge theory

4.1. *Solution of the unitarity constraints*

In §§ 2 and 3 we have seen how to write the Langevin equation for the unitary matrices U(t) starting directly from the group manifolds for U(1), U(2) and SU(2). Here we show that the stochastic quantisation can be easily done for any N by imposing the unitarity constraint by means of Lagrange multipliers. We consider the system in any number of dimensions, in this case the Wilson action is

$$S = -\frac{\beta}{2}\sum_P (\mathrm{Tr}\, U_P + \mathrm{Tr}\, U_P^+) \tag{4.1}$$

and the equation for the evolution of the element $U_{ij}(l, t)$ of the matrix $U(l, t)$ attached to the link l is

$$\frac{\partial U_{ij}(l, t)}{\partial t} = -\frac{\delta S}{\delta U_{ij}^*(l, t)} + [U(l, t)\lambda(l, t)]_{ij} + [\eta(l, t)U(l, t)]_{ij}. \tag{4.2}$$

Here we considered the $U_{ij}(l, t)$ as independent variables but we introduced N^2 Lagrange multipliers which guarantee the constraints

$$\lambda_{kl}\left(\sum_s U_{sk}U_{sl}^* - \delta_{kl}\right) = 0. \tag{4.3}$$

(We dropped the link index l and the time dependence; we shall put them back when necessary.)

At this point the stochastic force η is an arbitrary matrix and the integration measure is given by

$$D\eta(l, t) = \prod_t \prod_{i,j} \mathrm{d}\,\mathrm{Re}\,\eta_{ij}(l, t)\,\mathrm{d}\,\mathrm{Im}\,\eta_{ij}(l, t)\exp\left(-\tfrac{1}{2}\int_0 |\eta_{ij}(l, t)|^2\,\mathrm{d}t'\right). \tag{4.4}$$

To determine the λ_{jk}'s we use the condition

$$\sum_j \frac{\partial}{\partial t}(U_{ji}^* U_{jk}) = 0 \tag{4.5}$$

which together with equation (4.2) yields

$$2\lambda_{ij} = -\frac{\beta}{2}\sum_{\{P_l\}}[(U_{P_l})_{ik} + (U_{P_l}^+)_{ik}] - [(U^+\eta U)_{ik} + (U^+\eta^+ U)_{ik}]. \tag{4.6}$$

Replacing in (4.2) we obtain a matrix equation for the stochastic quantisation of the U(N) lattice gauge theory

$$\frac{\partial U(l,t)}{\partial t} = \frac{\beta}{4}\sum_{\{P_l\}} U(l,t)[U_{P_l}^+(t) - U_{P_l}(t)] + \mathrm{i}H(l,t)U(l,t) \tag{4.7}$$

where $H(l, t)$ happens to be a Hermitian matrix defined by

$$\mathrm{i}H(l,t) = \tfrac{1}{2}(\eta - \eta^+). \tag{4.8a}$$

Notice that $H(l, t)$ follows a Gaussian distribution with the same variance as $\eta(l, t)$. In particular

$$\langle H_{ik}(t_1)H_{ln}(t_2)\rangle_H = \delta_{in}\delta_{kl}\delta(t_1 - t_2). \tag{4.8b}$$

Let us emphasise that since U satisfies (4.5) it will be a unitary matrix at any time if it is at $t=0$.

Comparing (4.7) for $N=1$ with (2.10) we see that they differ in a factor $\frac{1}{2}$ in the term proportional to β. However also the variances of the Gaussian distributions (2.6) and (4.4) are different. It is immediate to check that if we rescale simultaneously the stochastic force and the time by factors $\frac{1}{2}$ and 2 respectively then both equations become equal, and the same happens for the variances. Since we are interested in the limit $t \to \infty$ the rescaling in t does not have any consequence. Similar considerations should be made to compare with the results of § 3.

4.2. *Langevin equations for the Wilson loops and strong coupling expansions*

As we did for the U(1) case we can obtain a Langevin equation for the Wilson loops. However now we have to be careful because the loop contains a path ordering

$$W_c(t) = \mathrm{Tr}[U(l_1,t)\ldots U(l_n,t)U(l_{n+1},t)\ldots U(l_L,t)]. \tag{4.9}$$

Its Langevin equation is easily seen to be

$$\frac{\partial W_c(t)}{\partial t} = \frac{\beta}{4}\sum_{\substack{l\in C\\ \{P_l\}}}[W_{c_1}(l,t) - W_{c_2}(l,t)] + \mathrm{i}\sum_{l\in C}\rho(l,t) \tag{4.10}$$

where

$$\begin{aligned} W_{c_1}(l_n,t) &= \mathrm{Tr}[U(l_1,t)\ldots U(l_n,t)U_{P_{l_n}}^+(t)U(l_{n+1},t)\ldots U(l_L,t)] \\ W_{c_2}(l_n,t) &= \mathrm{Tr}[U(l_1,t)\ldots U(l_n,t)U_{P_{l_n}}(t)U(l_{n+1},t)\ldots U(l_L,t)] \end{aligned} \tag{4.11}$$

and

$$\rho(l_n,t) = \mathrm{Tr}[U(l_1,t)\ldots H(l_n,t)U(l_n,t)\ldots U(l_L,t)]. \tag{4.12}$$

The loops C_1 and C_2 are defined by the sequences of links given in (4.11).

Because of the non-commutativity of the stochastic forces at different links and times the perturbative solution of (4.10) is a little more involved than in the Abelian case. The first order is

$$W_c^{(0)}(t) = \mathrm{Tr}\left(\prod_{l \in C} U^{(0)}(l, t)\right) \tag{4.13}$$

where

$$U^{(0)}(l, t) = T\left[\exp\left(\mathrm{i}\int_0^t H(l, t')\,\mathrm{d}t'\right)\right] \tag{4.14}$$

with T a time ordering operator. To evaluate the next orders it is not convenient to factor out a function $V_c(t)$ as we did in (2.13). Instead it is better to construct a perturbation series for each link. Then we propose

$$W_c(t) = \mathrm{Tr}\left[\prod_{l \in C} (U^0(l, t)V(l, t)\right] \tag{4.15}$$

where

$$V(l, t) = \sum_{n=0}^{\infty} (\beta/4)^n v^{(n)}(l, t) \tag{4.16}$$

and $v^{(0)}(l, t) = 1$.

To evaluate the $v^{(n)}(l, t)$ we replace $U(l, t) = U^{(0)}(l, t)V(l, t)$ in (4.7) and solving for each order we have

$$v^{(n)}(l, t) = \sum_{k=0}^{n-1} \int_0^t \mathrm{d}t'\, v^{(k)}(l, t') \sum_{\{P_l\}} [U_{P_l}^{(n-1-k)+} - U_{P_l}^{(n-1-k)}(t')]. \tag{4.17}$$

As an example of the strong coupling expansion of a Wilson loop we calculate the mean value of a plaquette up to order β in the limit $t \to \infty$.

From (4.15)

$$\begin{aligned}\mathrm{Tr}\, U_P(t) &\simeq \mathrm{Tr} \prod_{l \in P} U^0(l, t)[1 + (\beta/4)v^{(1)}(l, t)] \\ &\simeq \mathrm{Tr}(U_P^{(0)}(t)) + (\beta/4)\,\mathrm{Tr}[U_1^{(0)}v_1^{(1)}U_2^{(0)}U_3^{(0)}U_4^{(0)} \\ &\quad + U_1^{(0)}U_2^{(0)}v_2^{(1)}U_3^{(0)}U_4^{(0)} + U_1^{(0)}U_2^{(0)}U_3^{(0)}v_3^{(1)}U_4^{(0)} + U_P^{(0)}v_4^{(1)}](t)\end{aligned} \tag{4.18}$$

where 1, 2, 3, 4 stand for the links of the plaquette P.

The $v_l^{(1)}$ are given by (4.17)

$$v_l^{(1)}(t) = \sum_{\{P_l\}} \int_0^t [U_{P_l}^{(0)+}(t') - U_{P_l}^{(0)}(t')]\,\mathrm{d}t'. \tag{4.19}$$

To calculate $\mathrm{Tr}\, U_P(t)_H$ we have to average each order in β over all Hermitian matrices. In appendix 2 we show a possible way to do it; in the first order we have (see (A2.17))

$$\begin{aligned}\langle \mathrm{Tr}(U_P^{(0)}(t))\rangle_H &= N\,\mathrm{e}^{-2Nt} \\ &\to 0 \qquad \text{as } t \to \infty.\end{aligned} \tag{4.20}$$

As in the U(1) case it is possible to obtain a non-vanishing contribution in order β by picking up the plaquette P from the set $\{P_l\}$ for each $v_l^{(1)}(t)$. Noticing that

$U^{(0)}(l,t)U^{(0)+}(l,t')=U^{(0)}(l,t-t')$ we finally have

$$\langle \mathrm{Tr}\, U_P^{(1)}(t)\rangle_H = N\beta \int_0^t \exp[-2N(t-t')]\,\mathrm{d}t' \rightarrow \beta/2 \qquad \text{as } t\rightarrow\infty \tag{4.21}$$

and we see that in the large time limit it coincides with the well known result from the standard functional integral quantisation (Wilson 1974).

4.3. Connection with the Schwinger-Dyson equation

Similar to what we did in § 2 for the Abelian system this connection is established by showing that

$$\langle \rho(l,t)\rangle_H = \tfrac{1}{2}\mathrm{i}N\langle W_c(t)\rangle_H \tag{4.22}$$

holds at all orders in the strong coupling expansion.

First let us remark that because of the properties of this expansion that we have already found in (4.2), the general form of a typical term is still given by an expression like (2.22). Of course because of the non-Abelian character of the group there will be slight differences but they are not relevant for our arguments.

As a consequence we see that by comparing (4.9) and (4.12) the two averages in (4.22) differ only in the integral over the matrix $H(l,t)$. More explicitly we only have to prove that

$$\int \mathrm{D}H(l,t)(H(l,t)U^{(0)}(l,t))_{ij} = \tfrac{1}{2}\mathrm{i}N \int \mathrm{D}H(l,t)(U^{(0)}(l,t))_{ij}. \tag{4.23}$$

This can be done by using

$$(U^{(0)}(l,t))_{ij} = \left\{ T\left[\exp\left(\mathrm{i}\int_0^t H(l,t')\,\mathrm{d}t'\right)\right]\right\}_{ij} = \delta_{ij} + \mathrm{i}\int_0^t H(l,t')\,\mathrm{d}t' + \mathrm{i}^2 \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2\, H(l,t_1)H(l,t_2) + \ldots \tag{4.24}$$

and the Wick's theorem for the Gaussian expectation values.

Finally replacing (4.22) in (4.10) and taking the limit $t\rightarrow\infty$ we have

$$LN \lim_{t\rightarrow\infty} \langle W_c(t)\rangle_H = \frac{\beta}{2} \lim_{t\rightarrow\infty} \sum_{\substack{l\in C\\ \{P_l\}}} [\langle W_{c_1}(l,t)\rangle_H - \langle W_{c_2}(l,t)\rangle_H] \tag{4.25}$$

and we notice that this coincides with the Schwinger-Dyson equations for U(N) LGT (Foerster 1979, Eguchi 1979, Weingarten 1979). This allows us to identify

$$\left\langle \mathrm{Tr} \prod_{l\in C} U_l \right\rangle_S = \lim_{t\rightarrow\infty} \langle W_c(t)\rangle_H. \tag{4.26}$$

If the loop C is such that a given link l is traversed more than once this equation has to be modified (Foerster 1979, Eguchi 1979, Weingarten 1979). Although the algebra is rather lengthy it is possible to show that the Langevin equation we gave before is also in agreement with the Schwinger-Dyson equation for this kind of loop. As an example we consider the case where the link l_n is traversed twice in the same direction. Shortening the notation in an obvious way

$$W_c(t)=\mathrm{Tr}[U_1\dots U_{n-1}U_nU_{n+1}\dots U_{i-1}U_nU_{i+1}\dots U_L](t) \tag{4.27}$$

and after straightforward calculations similar to those we did for the simpler loops one sees that (4.22) is now replaced by

$$\langle\rho(l_n,t)\rangle_H=\tfrac{1}{2}\mathrm{i}[NW_c(t)_H+W_{c_1'}(l_n,t)W_{c_2'}(l_n,t)_H] \tag{4.28}$$

with

$$\begin{aligned}W_{c_1'}(l_n,t)&=\mathrm{Tr}[U_1\dots U_{n-1}U_nU_{i+1}\dots U_1](t)\\ W_{c_2'}(l_n,t)&=\mathrm{Tr}[U_{n+1}\dots U_{i-1}U_n](t).\end{aligned} \tag{4.29}$$

As before to prove this equality requires the evaluation of integrals only over $H(l_n,t)$. In particular integrals like

$$\int DH(l,t)(H(l,t)U^{(0)}(l,t))_{jk}(U^0(l,t))_{st} \tag{4.30}$$

and

$$\int DH(l,t)(U^{(0)}(l,t))_{jk}(U^{(0)}(l,t))_{st}$$

have to be calculated.

From (4.27) and (4.10) in the limit $t\to\infty$ we regain the Schwinger–Dyson equation for this loop (Foerster 1979, Eguchi 1979, Weingarten 1979)

$$\begin{aligned}LN\lim_{t\to\infty}\langle W_c(t)\rangle_H&=-\lim_{t\to\infty}\sum_{l\in C}\langle W_{c_1'}(l,t)W_{c_2'}(l,t)\rangle_H\\ &\quad+\frac{\beta}{2}\lim_{t\to\infty}\sum_{\substack{l\in C\\ \{P_l\}}}[\langle W_{c_1}(l,t)\rangle_H-\langle W_{c_2}(l,t)\rangle_H].\end{aligned} \tag{4.31}$$

5. SU(N) lattice gauge theory

The Langevin equations for SU(N) lattice gauge theory can be derived from the results of § 4.

As we saw in § 3 the only difference between the stochastic quantisation of U(2) and SU(2) is that in the last case we have to average only over Hermitian matrices of zero trace. However this is not enough for any N. To see this let us first notice that we want the condition

$$\partial\det U(l,t)/\partial t=0 \tag{5.1}$$

to hold at any time. If we now write

$$U(l,t)=\exp(\mathrm{i}K(l,t)) \tag{5.2}$$

then

$$\det U(l,t)=\exp(\mathrm{i}\,\mathrm{Tr}\,K(l,t)) \tag{5.3}$$

and if (4.7) were valid for any SU(N) with the only restriction Tr $H=0$ we would have

$$\begin{aligned}\partial \det U(l,t)/\partial t &= \mathrm{i} \det U \operatorname{Tr} \dot{K}(l,t) \\ &= \det U(l,t)\{\mathrm{i} \operatorname{Tr} H(l,t) + (\beta/4)[\operatorname{Tr} U^{+}(l,t) - \operatorname{Tr} U(l,t)]\}\end{aligned} \tag{5.4}$$

from where we see that condition (5.1) is not fulfilled. In order to get rid of the last term on the right-hand side we have to modify (4.7) for the link variable. The correct one for SU(N) lattice gauge theory is obtained by adding to it the term

$$\frac{\beta}{4N}\sum_{\{P_l\}} U(l,t)[\operatorname{Tr} U_{P_l}(t) - \operatorname{Tr} U^{+}_{P_l}(t)]. \tag{5.5}$$

Then the Langevin equation for the Wilson loop is

$$\begin{aligned}\frac{\partial W_c(t)}{\partial t} = \frac{\beta}{4}\sum_{\substack{l\in C\\ \{P_l\}}} & W_{c_1}(l,t) - W_{c_2}(l,t)] \\ & - \frac{\beta}{4N}\sum_{\substack{l\in C\\ \{P_l\}}} W_c(t) \operatorname{Tr}[U^{+}_{P_l}(t) - U_{P_l}(t)] + \mathrm{i}\sum_{l\in C} \rho(l,t).\end{aligned} \tag{5.6}$$

Let us emphasise that this term does not appear for SU(2); the trace of a matrix belonging to this group is real and the term vanishes.

The proof that when averaged over the random forces (5.6) yields in the limit $t\to\infty$ the correct Schwinger-Dyson equation can be done along the same lines for the group U(N).

For the sake of completeness we give in appendix 3 the Schwinger-Dyson equations for SU(N) lattice gauge theory.

As we already said the average has to be taken only over Hermitian matrices of zero trace. As a consequence (4.8b) has to be changed for

$$\langle H_{ik}(t_1) H_{ln}(t_2)\rangle_H = \delta(t_1 - t_2)(\delta_{in}\delta_{kl} - \delta_{ik}\delta_{ln}/N). \tag{5.7}$$

From this and the Wick decomposition we have

$$\langle \rho(l_n, t)\rangle_H = \mathrm{i}[(N^2-1)/2N]\langle W_c(t)\rangle_H \tag{5.8}$$

that together with (5.6) gives

$$\begin{aligned}\lim_{t\to\infty} L\frac{(N^2-1)}{N}\langle W_c(t)\rangle_H &= \beta \lim_{t\to\infty}\sum_{\substack{l\in C\\ \{P_l\}}} [\langle W_{c_1}(l,t)\rangle_H - \langle W_{c_2}(l,t)\rangle_H] \\ &\quad - \frac{\beta}{N}\lim_{t\to\infty}\sum_{\substack{l\in C\\ \{P_l\}}} [\langle W_c(t) \operatorname{Tr} U^{+}_{P_l}(t)\rangle_H - \langle W_c(t) \operatorname{Tr} U_{P_l}(t)\rangle_H]\end{aligned} \tag{5.9}$$

which agrees with (A3.3).

Equation (5.9) is not the right one when one link of the loop is traversed more than once. We have again checked the case where the link l_n is traversed twice in the same direction. Now equation (5.6) does not hold and has to be replaced by

$$\langle \rho(l_n, t)\rangle_H = \tfrac{1}{2}\mathrm{i}\{[(N^2-2)/N]\langle W_c(t)\rangle_H + \langle W_{c_1'}(l_n,t) W_{c_2'}(l_n,t)\rangle_H\} \tag{5.10}$$

and the corresponding Langevin equation becomes in the limit $t \to \infty$

$$\lim_{t\to\infty} L\frac{(N^2-2)}{N}\langle W_c(t)\rangle_H = \beta \lim_{t\to\infty} \sum_{\substack{l\in C\\ \{P_l\}}} [\langle W_{c_1}(l,t)\rangle_H - \langle W_{c_2}(l,t)\rangle_H]$$

$$-\lim_{t\to\infty}\sum_{l\in C}\langle W_{c_1'}(l,t)\,W_{c_2'}(l,t)\rangle_H - \frac{\beta}{N}\lim_{t\to\infty}\sum_{\substack{l\in C\\ \{P_l\}}}[\langle W_{c'}(t)\,\mathrm{Tr}\,U^+_{P_l}\rangle_H$$

$$-\langle W_c(t)\,\mathrm{Tr}\,U_{P_l}\rangle_H]$$

which is the Schwinger–Dyson equation for this case (see appendix 3).

The proof we have developed for the stochastic quantisation of Abelian and non-Abelian LGT can in principle be also applied to continuum quantum field theories. In the case of a scalar field this would be a much simpler proof than the usual diagrammatic one (Parisi and Wu 1981, Grimus and Hüffel 1982).

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Appendix 1. Change of variables for the Langevin equation

For completeness we present here the transformation properties of a coupled set of Langevin equations under a general change of variables (Graham 1977).

This set of equations is of the form

$$\dot{q}^\nu = f^\nu(q) + g_i^\nu(q)\xi^i(t) \tag{A1.1}$$

where $\nu = 1, \ldots, n$ numbers the macroscopic stochastic variables q^ν and $i = 1, \ldots, m$ the stochastic (Gaussian) forces ξ^i. $f^\nu(q)$ and $g^\nu(q)$ are functions of the coordinates q^ν.

Under a general change of variables

$$\mathrm{d}q'^\nu = \Lambda^\nu{}_\mu\,\mathrm{d}q^u \tag{A1.2}$$

where

$$\Lambda^\nu{}_\mu = \partial q'^\nu/\partial q^\mu \tag{A1.3}$$

the quantities q^ν, f^ν and g_i^ν have contravariant vector transformation properties. The Langevin equation given in (A1.1) is already written in a covariant form. To raise and lower indices it is possible to define a metric tensor (Graham 1977)

$$Q^{\nu\mu}(q) = g_i^\nu(q)g_k^\mu(q)\delta^{ik}. \tag{A1.4}$$

For instance the metric tensor obtained after the change of variables (3.4a)-(3.4b) is

$$Q^{\nu\mu} = \begin{pmatrix} 1 & & & & & \\ & 1/a^2 & & & 0 & \\ & & 1 & & & \\ & & & 1/(r\sin\theta_1\sin\theta_2)^2 & & \\ & & & & 1/(r\sin\theta_2)^2 & \\ & & 0 & & & 1/r^2 \end{pmatrix}. \tag{A1.5}$$

(rows and columns are ordered in the sequence a, θ, r, ψ, θ_1, θ_2).

In general it is possible to use a different set of stochastic forces in the Langevin equations without changing the mean values of the macroscopic variables. Of course this transformation should leave the Fokker-Planck equations unchanged. This equation can be written (Graham 1977)

$$\frac{\partial P(q,t)}{\partial t} = \left(-\frac{\partial}{\partial q^\nu}K^\nu(q) + \frac{\partial^2}{\partial q^\nu \partial q^\mu}Q^{\nu\mu}\right)P(q,t) \tag{A1.6}$$

where

$$K^\nu(q) = f^\nu(q) + (\partial g_i^\nu/\partial q^\mu) g_k^\mu \delta^{ik}. \tag{A1.7}$$

From here and (A1.4) we see that for this purpose it is enough to choose the new set of g_i^ν's (the coefficients of the stochastic forces) such that the metric tensor is kept unaltered and after that to observe in the forces $f^\nu(q)$ the variation in the $K^\nu(q)$ due to the new g_i^ν.

In order to prove the equivalence between the equations (3.14) and (3.15) and (3.18) for the unitary matrix we should notice that the first give the same Fokker-Planck equation as the set

$$\begin{aligned} \dot\theta &= -2\beta\cos\theta_2\sin\theta + h_+(t) \\ \dot\psi &= -h_-(t) + h_3(t)\left(\frac{\cos\psi\cos\theta_1 - \sin\psi\cot\theta_2}{\sin\theta_1}\right) \\ &\quad + h_4(t)\left(\frac{-\sin\psi\cos\theta_1 - \cos\psi\cot\theta_2}{\sin\theta_1}\right) \\ \dot\theta_1 &= -h_-(t)\sin\theta_1\cot\theta_2 + h_3(t)(\sin\psi + \cos\psi\cos\theta_1\cot\theta_2) \\ &\quad + h_4(t)(\cos\psi - \sin\psi\cos\theta_1\cot\theta_2) \\ \dot\theta_2 &= -2\beta\cos\theta\sin\theta_2 + h_-(t)\cos\theta_1 + h_3(t)\cos\psi\sin\theta_1 - h_4(t)\sin\psi\sin\theta_1 \end{aligned} \tag{A1.8}$$

where h_+, h_-, h_3 and h_4 have the same variance as η_θ, η_ψ, η_{θ_1} and η_{θ_2}. Actually by comparing the equations for θ we see that h_+ coincides with η_θ. It is a trivial exercise to check that the metric tensor associated with these equations is equal to the one given in (A1.5) once we impose the constraint $a(t) = r(t) = 1$. The four stochastic forces in (A1.8) can be taken to define a 2×2 Hermitian matrix. We first write

$$\begin{aligned} \eta_\theta &\equiv h_+ = (h_1 + h_2)/2 \\ h_- &= (h_1 - h_2)/2 \end{aligned} \tag{A1.9}$$

and the Hermitian matrix is

$$H=\begin{pmatrix} h_1 & h_3+\mathrm{i}h_4 \\ h_3-\mathrm{i}h_4 & h_2 \end{pmatrix}. \tag{A1.10}$$

From here we see that

$$\eta_\theta = \mathrm{Tr}\, H/2. \tag{A1.11}$$

Since

$$\begin{aligned}\mathrm{Tr}\, H^2 &= h_1^2+h_2^2+2(h_3^2+h_4^2)\\ &= 2(h_+^2+h_-^2+h_3^2+h_4^2)\end{aligned} \tag{A1.12}$$

the measure we have to use to average over Hermitian matrices is

$$\mathrm{D}H(t)=\prod_t \mathrm{d}h_1(t)\,\mathrm{d}h_2(t)\,\mathrm{d}h_3(t)\,\mathrm{d}h_4(t)\exp\left(-\tfrac{1}{8}\int_0^\infty \mathrm{Tr}\, H^2(t')\,\mathrm{d}t'\right). \tag{A1.13}$$

Appendix 2. Evaluation of an integral

We show here how to evaluate integrals of the form

$$I_{ij}(t)=\left\langle\left[T\exp\left(\mathrm{i}\int_0^t H(t')\,\mathrm{d}t'\right)\right]_{jk}\right\rangle_H \tag{A2.1}$$

which appear in the strong coupling expansion of the Langevin equations (4.7) and (4.10). An alternative procedure will be used in the last part of § 4 and in § 5; the advantage of the present calculation is that it does not require an expansion in powers of t.

We begin by noticing that if the interval $[0, t]$ is divided in N parts of size $\varepsilon\to 0$ such that $t=N\varepsilon$ is kept fixed then (A2.1) becomes

$$I_{jk}(t)=\lim_{\varepsilon\to 0}\langle(\exp[\mathrm{i}H(t)\varepsilon]\exp[\mathrm{i}H(t-\varepsilon)\varepsilon\ldots \mathrm{i}H(\varepsilon)\varepsilon])_{jk}\rangle_H. \tag{A2.2}$$

So we only need to evaluate

$$I_{lm}(\varepsilon)=\frac{1}{z}\int \mathrm{d}^{N^2}H\exp(-\varepsilon/a\,\mathrm{Tr}\, H^2)(\mathrm{e}^{\mathrm{i}\varepsilon H})_{lm} \tag{A2.3}$$

H is an $N\times N$ Hermitian matrix and the measure is given by

$$\mathrm{d}^{N^2}H=\prod_{i=1}^{N}\mathrm{d}H_{ii}\prod_{ij}\mathrm{d}(\mathrm{Re}\, H_{ij})\,\mathrm{d}(\mathrm{Im}\, H_{ij}) \tag{A2.4}$$

and

$$z=\int \mathrm{d}^{N^2}H\exp[-(\varepsilon/a)\,\mathrm{Tr}\, H^2]. \tag{A2.5}$$

Following Gross and Witten (1980) we diagonalise H by means of a unitary matrix V

$$H^{\mathrm{D}}=VHV^{-1} \tag{A2.6}$$

and change variables to the N eigenvalues λ_s and N^2-N parameters of the matrix V. The measure now reads

$$\mathrm{d}^{N^2}H=\mathrm{d}V\,\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N\prod_{1\leq p}\prod_{n\leq N}(\lambda_p-\lambda_n)^2 \tag{A2.7}$$

where $\mathrm{d}V$ is the invariant measure in $\mathrm{U}(N)$.

Since in terms of the new variables

$$(\mathrm{e}^{\mathrm{i}\varepsilon H})_{lm}=\sum_{s=1}^{N}V_{sl}^{*}V_{sm}\,\mathrm{e}^{\mathrm{i}\lambda_s} \tag{A2.8}$$

the integration over V can be done by using the orthogonality relations for the group $\mathrm{U}(N)$ and we obtain

$$I_{lm}(\varepsilon)=\delta_{lm}\frac{1}{Z}\sum_{s=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N\prod_{pn}(\lambda_p-\lambda_n)^2\,\mathrm{e}^{-\varepsilon/a}\sum_{q=1}^{N}\lambda_q^2\,\mathrm{e}^{\mathrm{i}\varepsilon\lambda_s}. \tag{A2.9}$$

The result of integrating over all the eigenvalues λ_j except $j=s$ is (Mehta 1967)

$$I_{lm}(\varepsilon)=\frac{\delta_{lm}}{zN}\sum_{s=1}^{N}\sum_{i=0}^{N=1}\int_{-\infty}^{\infty}\mathrm{d}\lambda_s\,\mathrm{e}^{\mathrm{i}\varepsilon\lambda_s}\psi_i^2(\sqrt{\varepsilon}/a\lambda_s) \tag{A2.10}$$

where

$$\psi_n(x)=(2^n n!\sqrt{\Pi})^{-1/2}\,\mathrm{e}^{-x^2/2}H_n(x) \tag{A2.11}$$

are the harmonic oscillator wavefunctions ($H_n(x)$ are the Hermite polynomials).

Doing similar manipulations with z we have

$$z=\sum_{i=0}^{N-1}\int_{-\infty}^{\infty}\mathrm{d}\lambda_S\,\psi_i^2(\sqrt{\varepsilon}/a\lambda_S) \tag{A2.12}$$

and, after integrating over λ_s in (A2.10) and (A2.12)

$$I_{lm}(\varepsilon)=\delta_{lm}\exp(-a\varepsilon/4)\frac{L_{N-1}^{(1)}(a\varepsilon/2)}{L_{N-1}^{(1)}(0)} \tag{A2.13}$$

where the $L_N^{(1)}(x)$ are the Laguerre polynomials (Gradshteyn and Ryzhik 1965). Finally up to order ε, $I_{lm}(\varepsilon)$ can be written as

$$I_{lm}(\varepsilon)=\delta_{lm}\exp(-Na\varepsilon/4) \tag{A2.14}$$

and combining this result with (A2.2)

$$I_{jk}(t)=\delta_{jk}\exp(-Na/4t). \tag{A2.15}$$

The integral (A2.1) is first order in the strong coupling expansion of the average of the link variable, i.e.

$$I_{jk}(t)=\langle U_{jk}^{(0)}(t)\rangle_H \tag{A2.16}$$

and as $t\to\infty$ it vanishes. Combining (A2.15) for all the links of a loop $W_c(t)$ the first-order contribution is

$$\langle W_c^0(t)\rangle_H=N\exp(-LNa/4t) \tag{A2.17}$$

(we took a loop C such that any link is traversed only once).

Appendix 3. Schwinger–Dyson equation for SU(N) lattice gauge theory

Following Foerster (1979), Eguchi (1979) and Weingarten (1979) we start with the expression (T^j is a generator of the group)

$$\langle \mathrm{Tr}(U_1 U_2 \ldots T^j U_{lm} \ldots U_{l_L})\rangle_S, \tag{A3.1}$$

change the variable U_{ln} to $(1+\mathrm{i}\varepsilon T^j)U_{ln}$ and collecting terms proportional to ε and using

$$\begin{aligned} &\sum_{j=1}^{N^2-1} (T^j T^j)_{ab} = \frac{(N^2-1)}{N}\delta_{ab} \\ &\sum_{j=1}^{N^2-1} (T^j)_{ab}(T^j)_{cd} = \delta_{ad}\delta_{bc} - \frac{\delta_{ab}\delta_{cd}}{N} \end{aligned} \tag{A3.2}$$

we obtain

$$\frac{(N^2-1)}{N} W_c = \beta/2(W_{c_1} - W_{c_2}) - (\beta/2N)\left\langle\left(\mathrm{Tr}\prod_{i\in C} U_i\right)\mathrm{Tr}(U_p^+ - U_p)\right\rangle_S \tag{A3.3}$$

where

$$W_c = \left\langle \mathrm{Tr}\prod_{i\in C} U_i\right\rangle_S \tag{A3.4}$$

and similarly for W_{c_1} and W_{c_2}.

If a given link appears more than once in the loop C (A3.3) does not hold anymore. In particular if it appears twice in the same direction we have

$$\begin{aligned} \frac{(N^2-2)}{N} W_c = {} & (\beta/2)(W_{c_1} - W_{c_2}) - (\beta/2N)\left\langle\left(\mathrm{Tr}\prod_{i\in C} U_i\right)\mathrm{Tr}(U_p^+ - U_p)\right\rangle_S \\ & - \left\langle\left(\mathrm{Tr}\prod_{i\in C_1'} U_i\right)\left(\mathrm{Tr}\prod_{j\in C_2'} U_j\right)\right\rangle_S \end{aligned} \tag{A3.5}$$

with C_1' and C_2' defined in (4.29).

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